Utility Option Pricing Model (UOPM)
The Two-State Asset Price Model

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We will introduce option pricing in complete and incomplete markets by building a two-state asset price model for the option, the underlying share, and a risk-free, zero-coupon bond. Our goal will be to...

Part	Description
1	Introduce the two state asset price model.
2	Use the two-state model to price an option in a complete market.
3	Use the two-state model to price an option in an incomplete market.
4	Extend the two-state model to an infinite-state model and price an option in an incomplete market.

To that end we will work through the following hypothetical problem...

Our Hypothetical Problem

We are given the following model assumptions...

Table 1: Model Assumptions

Symbol	Definition	Value
S_0	Share price at time zero ($)	20.00
B_0	Risk-free bond price at time zero ($)	100.00
X_T	Option exercise price at time T ($)	22.50
α	Bond continuous-time risk-free rate (%)	4.15
μ	Share continous-time return drift (%)	7.50
σ	Share continous-time return volatility (%)	18.00
p	Probability that share price will increase (%)	50.00
$(1-p)$	Probability that share price will decrease (%)	50.00
T	Option term in years (#)	3.00

Question 1: What are random asset prices at time T?
Question 2: What are expected asset prices at time T?
Question 3: What is the continuous-time risk-adjusted equity discount rate?

Share Price Equations

We will define the variable S_T to be random share price at time T, the variable μ to be continuous-time return drift, the variable σ to be continuous-time return volaility, and the variable z to be a random variable with value one or negative one at time T. The equation for random share price at time T as a function of known share price at time zero is...

$$S_T = S_0 \operatorname{Exp}\left\{\mu T + \sigma\sqrt{T}\, z\right\} \text{ ...where... } z \in \{-1, +1\} \tag{1}$$

We will define the variable $S(U)$ to be share price at time T given that the random variable $z = 1$ and the variable $S(D)$ to be share price at time t given that the random variable $z = -1$. The equations for these two possible share prices at time t are..

$$S(U) = S_0 \operatorname{Exp}\left\{\mu T + \sigma\sqrt{T}\right\} \text{ ...and... } S(D) = S_0 \operatorname{Exp}\left\{\mu T - \sigma\sqrt{T}\right\} \tag{2}$$

We will define the variable p to be the probability that the random variable equals one at time T, and the variable $(1-p)$ to be the probability that the random variable equals negative one at time T. This statement in equation form is...

$$\text{Prob}\Big[z=1\Big]=p \ \text{...and...} \ \text{Prob}\Big[z=-1\Big]=(1-p) \tag{3}$$

Using Equations (2) and (3) above, the equation for expected share price at time T is...

$$\mathbb{E}\Big[S_T\Big]=p\,S(U)+(1-p)\,S(D) \tag{4}$$

We will define the variable κ to be the risk-adjusted equity discount rate. Using Equation (4) above, the equation for the continuous-time discount rate is...

$$\text{if... } S_0=\mathbb{E}\Big[S_T\Big]\,\text{Exp}\Big\{-\kappa\,T\Big\} \ \text{...then...} \ \kappa=-\ln\left(S_0\Big/\mathbb{E}\Big[S_T\Big]\right)\Big/T \tag{5}$$

Bond Price Equations

We will define the variable B_T to be the risk-free, zero-coupon bond price at time T and the variable α to be the continuous time risk-free interest rate. The equation for bond price at time T is...

$$B_T=B_0\,\text{Exp}\Big\{\alpha\,T\Big\} \tag{6}$$

We will define the variable $B(U)$ to be bond price at time T given that the random variable $z=1$ and the variable $B(D)$ to be bond price at time T given that the random variable $z=-1$. The equations for these two possible bond prices at time T are..

$$B(U)=B_T \ \text{...and...} \ B(D)=B_T \tag{7}$$

Using Equations (3) and (7) above, the equation for expected bond price at time T is...

$$\mathbb{E}\Big[B_T\Big]=p\,B(U)+(1-p)\,B(D) \tag{8}$$

Option Price Equations

We will define the variable X_T to be the option exercise price at time T. This statement in equation form is...

$$X_T=\text{Option exercise price} \tag{9}$$

We will define the variables $C(U)$ and $C(D)$ to be call option payoffs at time T given that share price increases or decreases, respectively, over the time interval $[0,T]$. The equations for call option payoffs are...

$$C(U)=\text{Max}\Big[S(U)-X_T,0\Big] \ \text{...and...} \ C(D)=\text{Max}\Big[S(D)-X_T,0\Big] \tag{10}$$

We will define the variables $P(U)$ and $P(D)$ to be put option payoffs at time T given that share price increases or decreases, respectively, over the time interval $[0,T]$. The equations for put option payoffs are...

$$P(U)=\text{Max}\Big[X_T-S(U),0\Big] \ \text{...and...} \ P(D)=\text{Max}\Big[X_T-S(D),0\Big] \tag{11}$$

We will define the variables C_T and P_T to be the call option and put option price, respectively, at time T. Using Equations (3), (10), and (11) above, the equation for expected option prices at time T is...

$$\mathbb{E}\Big[C_T\Big]=p\,C(U)+(1-p)\,C(D) \ \text{...and...} \ \mathbb{E}\Big[P_T\Big]=p\,P(U)+(1-p)\,P(D) \tag{12}$$

Answers To Our Hypothetical Problem

Question 1: What are random asset prices at time T?

Using Equations (2), (7), (10), and (11) above and the model assumptions in Table 1 above, random asset prices at time T are...

$$\begin{aligned} S(U) &= 20.00 \times \text{Exp}\left\{0.0750 \times 3.00 + 0.1800 \times \sqrt{3.00}\right\} = 34.21 \\ S(D) &= 20.00 \times \text{Exp}\left\{0.0750 \times 3.00 - 0.1800 \times \sqrt{3.00}\right\} = 18.34 \\ B(U) &= 100.00 \times \text{Exp}\left\{0.0415 \times 3.00\right\} = 113.26 \\ B(D) &= 100.00 \times \text{Exp}\left\{0.0415 \times 3.00\right\} = 113.26 \\ C(U) &= \text{Max}\left[34.21 - 22.50, 0\right] = 11.71 \\ C(D) &= \text{Max}\left[18.34 - 22.50, 0\right] = 0.00 \\ P(U) &= \text{Max}\left[22.50 - 34.21, 0\right] = 0.00 \\ P(D) &= \text{Max}\left[22.50 - 18.34, 0\right] = 4.16 \end{aligned} \tag{13}$$

Question 2: What are expected asset prices at time T?

Using Equations (4), (8), (12), and (13) above and the model assumptions in Table 1 above, expected asset prices at time T are...

$$\begin{aligned} \mathbb{E}[S_3] &= 0.50 \times 34.21 + (1 - 0.50) \times 18.34 = 26.27 \\ \mathbb{E}[B_3] &= 0.50 \times 113.26 + (1 - 0.50) \times 113.26 = 113.26 \\ \mathbb{E}[C_3] &= 0.50 \times 11.71 + (1 - 0.50) \times 0.00 = 5.86 \\ \mathbb{E}[P_3] &= 0.50 \times 0.00 + (1 - 0.50) \times 4.16 = 2.08 \end{aligned} \tag{14}$$

Question 3: What is the continuous-time risk-adjusted equity discount rate?

Using Equations (5) and (14) above and the model assumptions in Table 1 above, the equity discount rate is...

$$\kappa = -\ln\left(20.00 \Big/ 26.27\Big]\right) \Big/ 3.00 = 9.09\% \tag{15}$$